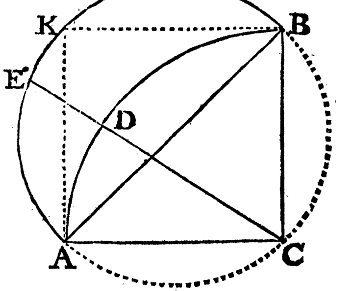
A Letter of Dr Wallis *to Dr* Sloan, *concerning the* Quadrature *of the* Parts *of the* Lunula *of* Hippocrates Chius, *performed by Mr* John Perks; *with the further* Improvements *of the same, by Dr* David Gregory, *and Mr* John Caſwell.

SIR,

THE *Squaring* a certain *Lunula* by *Hippocrates Chius* long ſince, hath been known (as to the whole *Lunula*) for many Ages. But (as to the *Parts* of it, and the *Appurtenances* thereunto,) *New* Diſcoveries have been lately made, which (I think) had not been conſider'd by any before this preſent Age.

I received (in *November* 1699.) from Mr. *John Perks* (Maſter of an Hoſpital at *Old-Swynford* in *Worceſter-ſhire*, founded by Mr, *Thomas Foley*) a brief account of his *Squariug* the *Portions* of *Hippocrates's Lunula*; with which (I preſume) you will not be diſpleaſed.

For the better underſtanding of which; I ſhall premiſe as known.(becauſe long ſince demonſtrated,) That, If on AB (the



Chord of ADB, the Quadrantal Arc of a Greater Circle, who's Center is C,) be deſcribed, as on a Diameter, a Semi-circle ABE;

 This

This Semi-circle, will be Equal to that Quadrant. (Because the Squares of their Diameters, are as 2 to 1 ; And, in such proportion are their respective Circles ; and therefore a Quarter of the one, equal to Half the other.)

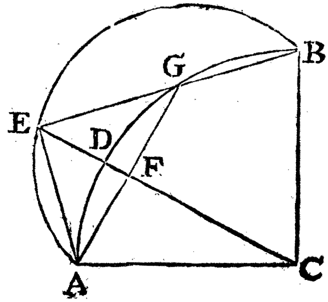
And, consequently, If, from each of these, we subtract the common Segment ABD : the *Remaining Lunula* ADBE (on the one side) will be Equal to the *Remaining Triangle* (on the other side) ABC. (Or, to ABK, supposing AB bisected in K ; that is, to half the Square CK, inscribed in the Lesser Circle.) Which is commonly called, *The Squaring of Hippocrates's Lunula* ; That is, the Finding a *Rectilinear Figure* (which may be easyly reduced to a Square) equal to that *Lunula*.

This being premised ; The Point in hand, is, the *Squaring* a given *Portion* of such *Lunula* : suppose ADE, cutt-off by a Streight Line CDE, drawn from the Center C. Which Mr *Perks* (not knowing that the like had been before attempted by any other) doth perform after this manner ; *viz.*

Drawing the Streight Lines EA, and EB (cutting the Arc EB in G,) and, on AG, a perpendicular EF, (which will therefore pass to the Center C, because Bisecting AG at Right-angles ;) *The Right-lined Triangle* AFE, *is equal to* ADE, *the proposed Portion of the Lunula.*

His Demonstration is to this purpose : *viz.*

ADB being a Quadrantal Arc ; the Angle AGB will be *Three Halves* of a Right Angle ; (and its Conjunct Angle EGA, *Half* a Right Angle.) And that Angle (being External to the Trian-



gle AGE,) is Equal to the Two Opposite Internals GEA + EAG. Whereof GEA (because an Angle in the Semicircle AEB), is a Right Angle ; and therefore EAG is *Half* a Right Angle, (as are also FEG, and FEA.) And the Three Triangles AFE, GFE,

and

and GEA, each of them *Half a Square*. And AG to AE, as $\sqrt{2}$ to 1 (proportional to the Respective *Radii* of the Two Circles.) And the Like Segments ADG, AE, in their Respective Circles (as the Squares of their Respective *Radii*) as 2 to 1. And therefore the Semi-segment AFD, equal to the Segment AE. And consequently (one taking from the Triangle as much as the other addes to it) the *Portion of the Lunula* ADE, equal to the *Triangle* AFE. Which was to be Demonstrated.

(I take the liberty (both in this and the things that follow) to vary somewhat from the Authors Words, (but to the same sense, and without any disadvantage to Them,) so as to Design the same Respective Points (in all the Figures) by the same Letters. Which makes it somewhat Shorter (without Repeating the same Construction anew for every Figure;) and prevents the Confusion which might arise to the Fansy, if the same Respective Points, in several Figures, were designed by different Letters; and the same Letters, in the different Figures, design different Points.)

If the Point E chance to be in K (the middle of the Arc AEB) there will be no Intersection at G (the Points G, B being then coincident, but without any disturbance to the Demonstration:) If it happen beyond it, toward B; then G will be on the other side; and what is here sayd of EGB, must be accommodated to EGA: which things are so obvious, as not to need any long discourse.

The whole proceeds upon the same general notion with that of squaring the whole *Lunula* (and some other Curve-lined Figures;) that, if as much be added to the one side, as is taken from the other, the Equality remains.

And the stress of the Demonstration, is, to prove the segments ADG and AE, to be *Like Segments*; and therefore Proportional to their Respective Circles; the Whole of one, equal to Half the other.

The Ground of the whole Process is plainly this, The Angle ACE, being an Angle at the Center of the Greater Circle, but at the Circumference of the Lesser, the line CDE (as it passeth from CA to CB) doth, in the same proportion, divide the Quadrantal Arc ADB, and the Semicircular AEB: whence all the rest doth naturally follow.

And this is Applicable to other *Lunula's* (beside that of *Hippocrates*) if (by altering the Angle at F, or otherwise,) we take-in such a Portion of the common Segment ABD on the one side (instead of AE cut-off on the other side) as the Proportion of the two Circles requires.

 I shewed

I shewed this Quadrature of Mr. *Perks* to Dr. *David Gregory* (our learned Professor of Astronomy at *Oxford*,) who gives his Opinion about it (with his Improvement of it) in a Letter of his to me; which I shall give you in his own words,

"Reverend Sir, The *Quadrature* of the *Parts* of the *Lunula* of "*Hippocrates Chius*, by Mr. *Perks* (which you shewed me) is "very Elegant.

"I remember, the like was done, some years since, by Monsieur "*Tchirnhause*; who assigns, as equal to the same Portion, not the "same Triangle with that of Mr. *Perks*, but another Equivalent "thereunto, (as I shall shew by and by.) We have his *Theorem*, "in the *Acta Lipsiæ*, for the Month of *September*, 1687. But, "without any *Demonstration*.

"But, both the One and the Other, seem not to have considered "this affair in its full extent.

"For, if you compleat the Two Circles, whose Arcs contain "the *Lunula* of *Hippocrates*; the same is true, as well of the Points "in the other Semi-circle ACB, as of those in the Semi-circle AEB; "and, for the same Reasons. As appears in the Scheme annexed, "wherein I have mark'd the Points in the Semi-circle ACB, (cor-"respondent to those of Mr. *Perks* in AEB,) with the correspon-"dent small Letters of the Roman and Greek Alphabets.

"If Mr. *Perks* had made his construction universal; by ma-"king both EA and EB, meet with the Greater Circle, (which he "might have done by protracting these Lines and the Greater "Circle 'till they meet;) he might have found that the Portions "of the Spaces A ε CM, BHCN, (supposing MCN parallel to AB) "are Quadrable as well as those of *Hippocrates*'s *Lunula*: And "that E A γ being a streight Line, the Portion AED of *Hip*-"*pocrates*'s *Lunula*, is to A ε δ (the Correspondent of A ε CM) "in the Duplicate Proportion of C ε to A ε. For E R ε (at R the "Center of the Lesser Circle) is, in this case, a Right Angle.

"Moreover; If you take any Point ε in the Semi-circle ACB, "and proceed according to Mr. *Perk*'s construction Universalized "as above-said; you will find, on the one side, the *Trilineum* "A ε δ (contained by the Arcs A ε, A δ, and the streight line ε δ) "equal to the Rectilineal Triangle A ε φ. And, on the other side, "the *Trilineum* contained by the Arc B ε (the Complement of ε A "to the Semi-circumference,) and the Arc B d (the Complement "of A δ to the Fourth part of the Circumference,) and the streight "line ε d, (that is, the *Trilineum* BHCd diminished by the Se-

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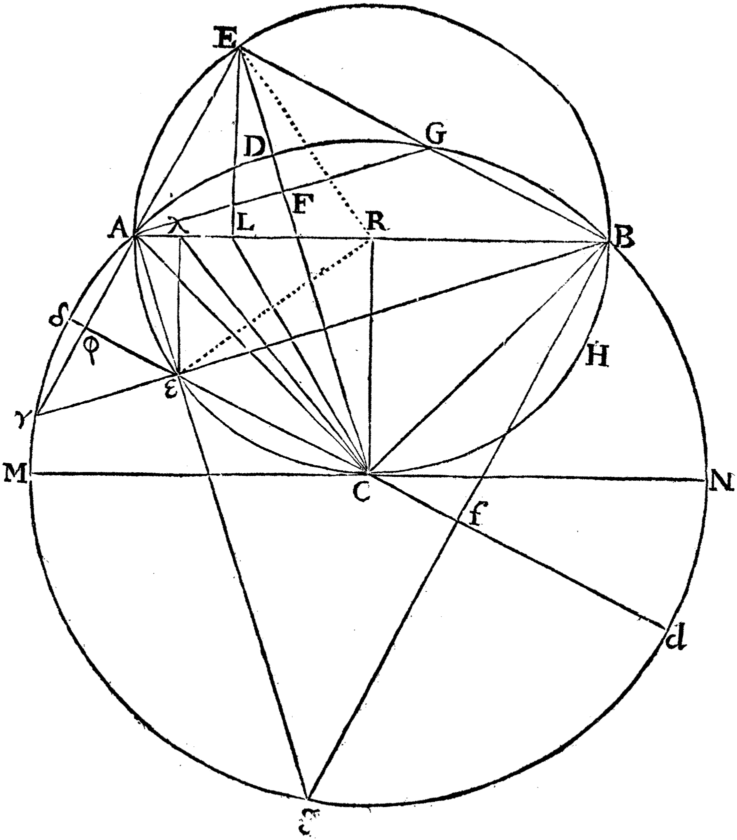


" gment Cε;) to be equal to the Rectilineal Triangle Bεf. And,
" that those two spaces Aεδ, and the *Difference* of BHCd from
" the Segment Cε (parts of the *Lunula* ACB g γ A) taken to-
" gether, are equal to the Triangle ACB; as well as the two
" Spaces AED and BED, parts of the *Lunula* of *Hippocrates*.

" So that, upon the whole, it appears, that the Two Circles
" (containing the *Lunula* of *Hippocrates*) being completed; this
" *Lunula* AEBGA, and the other ACB g γ A, make up one System,
" and are *Conjugate* Figures.

" For, (drawing a streight line CDE, or Cεδ, or Cεd, at pleasure
" through C the Center of the Greater Circle, and cutting those
" two Circles,) the Space contained within two Arcs of these two
" Circles and part of the said streight line, (as AED, or Aεδ, or
" BHεd,)

"BH ε d,) is equal to the Rectilineal Triangle AEF, or A ε φ, or "B ε f, respectively.

"And it so happens, that, if this line going out from C, be on "the same side of the Diameter MN with the *Lunula* of *Hippo-* "*crates*; the foresaid Space (which receives a perfect Quadra- "ture) is solitary; (such as are the Parts of *Hippocrates*'s *Lunula*; "and of the two Spaces A ε CM, BHCN; which therefore are Parts "of the *Lunula* more nearly relating to one another.)

"But if that Line going out from C, be on the other side of "MN; then the Space which is equal to the Rectilineal Triangle, "is, the *Difference* of two Mixtilineal Figures, (the one a Tri- "lineum, the other a Segment of the Lesser Circle,) as is above- "said; neither of which can be squared severally.

"All these particulars are plain from Mr. *Perks*'s Demonstra- "tion; which, with a little variation (such as is usual in the dif- "ferent *Cases* of the same *Theoreme*) is applicable to all of them: "though perhaps he was not aware of it.

"In the Dimension of the Parts of *Hippocrates*'s *Lunula*, "it might perhaps be expected, that the Triangle assigned equal to "a Portion of the *Lunula*, should be Part of the Triangle to "which that whole *Lunula* is wont to be assigned equal; (that is, "that the Triangle assigned equal to the Portion ADE, should be "the respective part of ACB which is equal to the whole *Lunula*;) "which in that of Mr. *Perks* is not.

"But, in that of Mr. *Tschirnhause* (above-mentioned) it is so, "which is to this purpose.

"If from any Point E, in the circumference of the Lesser Circle, "we let fall on AB, a Perpendicular cutting it in L, and draw the "line CL; the Triangle CAL, is equal to the Portion of the *Lu-* "*nula* AED. (And, consequently, the Triangle CBL, equal to "the Portion BED.)

"Which (because Mr *Tschirnhause* hath not at all done it) "I shall briefly Demonstrate, so as the Demonstration may reach "the *Portions* of the *Conjugate* Space ACB g γ A.

"For the Triangles ACB, AEF, are like Triangles, each being "the half of a Square: And therefore, by 19 el. 6, the Triangle "ACB is to the Triangle AEF in the duplicate proportion of BA "to AE, that is, by 8, el. 6, as BA is to AL. But, by 1. el. 6, the "Triangle ACB is to the Triangle ACL as BA is to AL. There- "fore, by 9. el. 5, the Triangles ACL and AEF are equal. But "the Triangle AEF is (by Mr *Perks*) proved equal to the Por-

"tion

" tion AED. And therefore the said Portion AED is also equal " to the Triangle ACL.

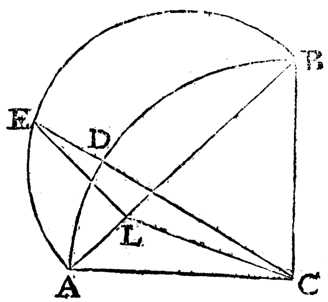
" I am, Sir, Your &c. D. *Gregory*.

Mr *Caswell* had a sight of this Quadrature of Mr *Perks* (before Dr *Gregorie* or I had seen it ;) And had given a *Specimen* of its being capable of further Improvement. But, without having Leisure, or giving himself the Trouble, of pursuing it through all its Appendages. I would (with his leave) have here inserted that *Specimen*: But he chose rather to decline it ; saying, He thought it needless, because Dr *Gregorie* had, since, done the like more fully.

The Result of it, is to this purpose ; On the Center B, he draws by A, a Third Circle ; which forms another *Lunula*, than that of *Hippocrates*: And he doth (very dextrously) Square the *Portions* of this *Lunula*. And doth thereby let us in, to a New System, which may be pursued in like manner as Dr *Gregorie* hath done that of *Hippocrates*.

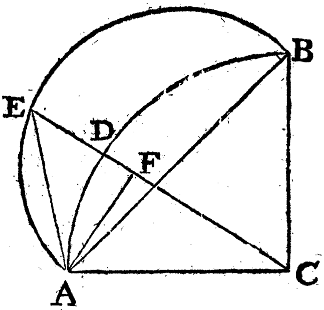
After these learned Disquisitions, on so trite a Subject ; it will not be needful for me to say much. I shall but briefly Compare the Two Quadratures of Mr *Tschirnhause* and Mr *Perks*, (wherein they Agree or Differ with each other.) And then shew, How, by either of them, to Divide the *Lunula* in any Given Proportion.

Monsieur *Tschirnhause* ; Letting fall, from E (on AB) a Perpendicular EL, determines the Triangle ALC equal to the Portion ADE.



Which being admitted ; We may thus Divide the *Lunula* in any Given Proportion. If we divide AB, at L, in such Given Proportion ; CL will, in the same proportion (because of the Common Altitude) divide the Triangle ACB (which is equal to the Whole *Lunula*.) And LE (erected at Right Angles on ALB) will determine the Point E ; from whence if we draw, to C, the Streight line EC, this will, at DE, divide the *Lunula* in the same Proportion.

Mr *Perks* ; On EDC, drawing the Perpendicular AF, determines the Semi-quadrate AFE, equal to the proposed Portion.



tion ADE. Which Semi-quadrate, is a *Like Figure*, and *a like situate* to AE, as is ACB to AB.

And therefore (because like Figures are in the Duplicate Proportion of their respective Sides) If we so inscribe AE, as that the Square of AE be to the Square of AB, in such Given Proportion, the *Lunula* will at DE, be so divided as is required.

And this will hold (if duly applied, according as the different Cases may require) though E be taken (in the Continuation of the Semi-circle) beyond B. For (still) Like Figures, will be in Duplicate Proportion of their Respective Sides; and $CE = CD \pm DE$. And the same is yet improveable much further.

I forbear to Apply this to the several Parts of the whole Systeme, considered by Dr *Gregorie*, (Or to that of Mr *Caswell*,) that I be not too Teadious.

Much less shall I give my self the trouble to consider the Solids to be made by the Conversion of it, or of its parts, about a given Axis, (as MN, or AB, or AC, or BC, &c.) with their Surfaces and Centers of Gravity; as I have done elsewhere for the *Cycloid*: But such as are at Leisure (and think it worth the while,) may do it by such like Methodes as I have made use of for the Cycloide,

I am *SIR*,

Yours to serve you,

JOHN WALLIS.

Post-script.

In the *Transactions* for the Month of *August* last past; *Numb.* 255. A Letter of mine, is very faultyly Printed. I desire that the *Errata* may be thus Corrected.

Pag. 280. *l.* 24. ut ait. *p.* 281. *l.* 15. differentias infinitesimas. *p.* 282. *l.* 12. (ut antea) rerum Novitas. *l.* 14. Messis. *l.* 15. Et quidem. *l.* 16. Atque hinc. *l.* 17. natura. *l.* 22. Academia. *l.* 25. reapse. *l.* 33. misi. *p.* 283. *l.* 5. desperatum. *l.* 11. Sueci. *l.* 17. itinere. *l.* 25. adornat. *l.* 33. Cœno. *p.* 284. *l.* 1. sita. *l.* 13. redeundo) sensim. *l.* 17. motibus. *l.* 19. penitius. *l.* 22. materiæ. *l.* 23. perpendiculum erectos) ad. *l.* 24. longo tractu. *l.* 25. præ se. *l.* 29. Multaque. *l.* 30. annos. *l.* 31. deprompta) mihi videntur huc. *l.* 32. aliò. *l.* 34. cœnosum, turbidum. *l.* 35. Isthmo. *l. ult.* *The Words* P. S. Aug. 29. 1699. *should have stood at lin.* 20. /

Numb. 257. *p.* 346. *l.* 11. the *Solar Tropical* year. *p.* 349. *l.* 2. suggested by. *p.* 351. *l.* 34. stands thus.